A PROOF OF RAABE'S TEST

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We give an alternative proof of one part of Raabe's test via summation by parts (aka Abel's lemma). We will prove the following:

Theorem 1 (Raabe's test, part 1)**.** *If (x_n) is a sequence of positive numbers, and there exists $a > 1$ such that*

$$\frac{x_{n+1}}{x_n} \leq 1 - \frac{a}{n} \tag{1}$$

for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} x_n$ is convergent.

Proof. For all $N \in \mathbb{N}$,

$$\begin{aligned}
\sum_{n=1}^{N} x_n &= \sum_{n=1}^{N} [(n+1) - n] x_n \\
&= \sum_{n=1}^{N} (n+1) x_n - \sum_{n=0}^{N-1} (n+1) x_{n+1} \\
&= (N+1) x_{N+1} - x_1 + \sum_{n=1}^{N-1} (n+1)(x_n - x_{n+1}) \\
&= (N+1) x_{N+1} - x_1 + \sum_{n=1}^{N-1} (n+1) \left(1 - \frac{x_{n+1}}{x_n}\right) x_n \\
&\geq (N+1) x_{N+1} - x_1 + \sum_{n=1}^{N-1} \frac{a(n+1)}{n} x_n.
\end{aligned}$$

Hence

$$\sum_{n=1}^{N-1} \left(\frac{a(n+1)}{n} - 1 \right) x_n \leq x_1 - (N+1) x_{N+1},$$

which implies

$$\sum_{n=1}^{N-1} (a-1) x_n \leq x_1,$$

or

$$\sum_{n=1}^{N-1} x_n \leq \frac{x_1}{a-1}$$

since $a > 1$. It follows that the partial sums of $\sum_{n=1}^{\infty} x_n$ are bounded above; since the series is a sum of non-negative numbers, this proves that $\sum_{n=1}^{\infty} x_n$ is convergent.

□

We remark that clearly the conclusion of the theorem would still hold, if (1) holds true only for all sufficiently large n.